

The canonical HANK model

Goethe Heterogeneous-Agent Macro Workshop

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Just saw: How to solve steady states and PE transitional dynamics of neoclassical heterogeneous agent models.

Next: Introducing “HANK”.

- 1 The canonical HANK model
- 2 Three instructive special cases
- 3 Solving the model using blocks and DAGs
- 4 Summary

The canonical HANK model

Introducing the canonical HANK model

- We now embed the standard incomplete markets model of consumption and saving into a New-Keynesian model
- Along the way, we will allow for a **government**: bonds, taxes, gov. spending
- Will mostly follow **Auclert et al. (2024)** (henceforth IKC), though allowing for monetary policy, too
- Will set up the model in the **sequence space**:
 - assume economy in steady state, feed in perfect foresight aggregate shocks at $t = 0$, study response thereafter (“MIT shocks”)
 - keep in mind **certainty equivalence** !!

- Household i solves

$$\begin{aligned} \max_{c_{it}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (u(c_{it}) - v(n_{it})) \\ & c_{it} + a_{it} \leq (1 + r_t)a_{it-1} + z_{it} \\ & a_{it} \geq \underline{a} \end{aligned}$$

- This is the “sequence” analogue of the Bellman equation from before

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- This is the “sequence” analogue of the Bellman equation from before
- Real after tax income is

$$z_{it} = (1 - \tau_t)y_{it} \qquad y_{it} = \frac{W_t}{P_t} e_{it} n_{it}$$

- Here, e_{it} is idiosyncratic productivity, normalized so that $\mathbb{E}_i[e_{it}] = 1$
- Can capture progressive taxation as in [Heathcote et al. \(2017\)](#) (see IKC paper)

Unions and sticky wages

- For our canonical HANK model, we'll work with **sticky wages** (not prices)
 - with sticky prices, can get countercyclical profits
- ... redistribution from wage to profit earners in recession... matters in HANK!
- ... strange results can happen (examples: [Bilbiie 2008](#), [Broer et al. 2020](#))

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- Microfound sticky wages extending [Erceg et al. \(2000\)](#) (see IKC)
 - + “labor allocation rule”: which agent works what fraction of total labor N_t ?
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- **Today:** will use simple ad-hoc wage Phillips curve (details won't matter)

$$\pi_t^w = \underbrace{\kappa \left(v'(N_t) - \frac{\epsilon - 1}{\epsilon} (1 - \tau_t) \frac{W_t}{P_t} u'(C_t) \right)}_{\text{wedge in labor FOC of "average" agent}} + \beta \pi_{t+1}^w$$

- Representative firm with aggregate production function, linear in labor

$$Y_t = X_t N_t$$

where X_t is TFP

- Assume **flexible prices** $\Rightarrow$

$$P_t = \frac{W_t}{X_t} \quad \Leftrightarrow \quad \frac{W_t}{P_t} = X_t$$

Real wage is exogenous. No profits!

- Goods inflation π_t = wage inflation π_t^w minus TFP growth

Government: Fiscal policy

Government sets fiscal policy, consisting of paths

- G_t of gov spending
- T_t of total tax revenue, controlled via τ_t

$$T_t = \tau_t Y_t$$

- B_t of government bonds, uniformly bounded (no Ponzi schemes)

subject to government budget constraint

$$B_t = (1 + r_t) B_{t-1} + G_t - T_t$$

Can set any two of those and the third follows.

Government: How is after tax income distributed?

- Total after-tax income is

$$Z_t \equiv Y_t - T_t = (1 - \tau_t) Y_t$$

- Because of linear taxes, we find for individual after-tax income

$$y_{it} = e_{it} Y_t \quad \Rightarrow \quad z_{it} = e_{it} Z_t$$

- z_{it} simply a share of total after-tax income Z_t . Will be convenient.

Government: Monetary policy

Monetary authority follows an interest rate rule. Allow for two kinds of rules:

- **standard Taylor rule.** (linearized)

$$i_t = r + \phi_\pi \pi_t + \epsilon_t$$

here: r = steady state real rate, ϵ_t = monetary shock

- **real rate rule.**

$$r_{t+1} = r + \epsilon_t \quad \Leftrightarrow \quad i_t = r + \pi_{t+1} + \epsilon_t$$

Equivalent to Taylor rule with coefficient 1 on expected inflation.

Note: π_{t+1} vs π_t not key (same in cts time!), key is $\phi_\pi = 1$

Why allow for “real rate rule”? Huge gain in tractability! All monetary policy acts via changing real rate. Cost is small if Phillips curve is flat (π_t moves little).

Definition of equilibrium

- All agents optimize and markets clear

$$G_t + C_t = Y_t$$

$$A_t = B_t$$

where household aggregates are

$$C_t = \int c_t^*(a_-, e) dD_t(a_-, e)$$

$$A_t = \int a_t^*(a_-, e) dD_t(a_-, e)$$

Computing the steady state

How can we find the steady state of this model?

1. Normalize $Y = 1$, calibrate r and B, G . Set $T = G + rB$.
2. Can use **same code** as for standard incomplete markets model:
 - instead of $e_{it}Y$ now use $e_{it} \cdot (Y - T)$
 - choose β to match $A = B$.
3. $G + C = Y$ holds by Walras law! Done!

Three instructive special cases

Special cases

Will introduce three special cases that are helpful to analyze and compare the HA model to.

1. Representative-agent model (RA) — [Woodford 2003, Galí 2008]
2. Two-agent model (TA) — [Campbell and Mankiw 1989, Galí et al. 2007, Bilbiie 2008]
3. Zero-liquidity model (ZL) — [Werning 2015, Ravn and Sterk 2017, Bilbiie 2021]

Only difference across models: how C_t is determined given real rates r_t and after-tax incomes Z_t .

Steady state aggregates are identical across models.

Representative-agent model

- This is the standard NK model (with wage rigidities)
- Consumption solves

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

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which has the solution

$$C_t = \frac{\beta^{t/\sigma} q_t^{-1/\sigma}}{\sum_{s \geq 0} \beta^{s/\sigma} q_s^{1-1/\sigma}} \left[\sum_{s \geq 0} q_s Z_s + (1 + r_0)a_{-1} \right]$$

where $q_t \equiv (1 + r_1)^{-1} \dots (1 + r_t)^{-1}$.

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where $q_t \equiv (1 + r_1)^{-1} \dots (1 + r_t)^{-1}$. With $r_t = r = \beta^{-1} - 1$, this is just

$$C_t = \frac{r}{1+r} \sum_{s \geq 0} (1+r)^{-s} Z_s + r a_{-1}$$

Two-agent model

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- Jointly pin down aggregate consumption

$$C_t = (1 - \mu) c_t^{PIH} + \mu c_t^{HTM}$$

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- Does that mean all Euler equations fail?

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- Eventually, **all** agents must have zero assets, so $c_{it} = z_{it} = e_{it}Z_t$
- Does that mean all Euler equations fail? Consider:

$$z_{it}^{-\sigma} \geq \beta(1+r_{t+1})\mathbb{E}_t[z_{it+1}^{-\sigma}] \quad \Leftrightarrow \quad z_t^{-\sigma} \geq \beta(1+r_{t+1}) \underbrace{\mathbb{E}\left[\frac{(e')^{-\sigma}}{e^{-\sigma}} \middle| e\right]}_{\rho(e)} z_{t+1}^{-\sigma}$$

- The last Euler equation to fail as we reduce β is that of $\bar{e} = \arg \max \rho(e)$
- Tractable! For instance, steady state r solves $\beta(1+r)\rho(\bar{e}) = 1$

Solving the model using blocks and DAGs

- Throughout this workshop, we will see that it is very useful to break models into “blocks”
- This language is often loosely used in practice, we will formally define them
 - reference is [Auclert et al. \(2021\)](#)
- We will write sequences of variables, e.g. $\{r_t\}$, as vectors $\mathbf{r} = (r_0, r_1, \dots)'$.

Defining blocks and models

Block: mapping from sequences of *inputs* to sequences of *outputs*.

Examples:

- Household block: $\mathbf{r}, \mathbf{Z} \rightarrow \mathbf{C}, \mathbf{A}$

$$V_t(e, a_-) = \max_{c, a} u(c) + \beta \mathbb{E}_t V_{t+1}(e', a)$$

$$\text{s.t. } c + a = (1 + r_t)a_- + e \cdot Z_t \quad a \geq \underline{a}$$

1. backward iteration to get policies $a_t(e, a_-), c_t(e, a_-)$.
2. forward iteration to get distribution $D_t(e, a_-)$.
3. aggregation: $A_t = \int a_t(e, a_-) dD_t(e, a_-)$, $C_t = \int c_t(e, a_-) dD_t(e, a_-)$.

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- Fiscal policy block: $\mathbf{r}, \mathbf{T}, \mathbf{G}, \mathbf{Y} \rightarrow \mathbf{B}, \mathbf{Z}$

$$B_t = G_t - T_t + (1 + r_t) B_{t-1}$$

$$Z_t = Y_t - T_t$$

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- Goods market clearing block: $\mathbf{Y}, \mathbf{C}, \mathbf{G} \rightarrow \mathbf{H} \equiv \mathbf{C} + \mathbf{G} - \mathbf{Y}$

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Examples:

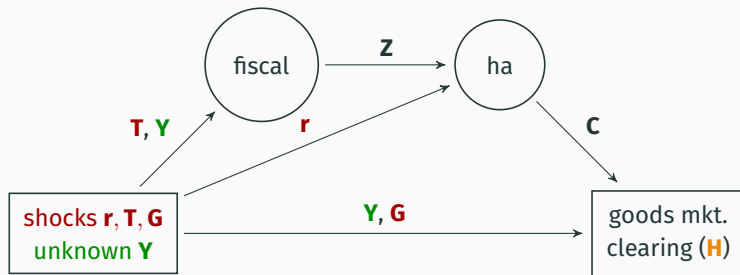
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Model: combination of blocks

- some inputs are exogenous **shocks**, e.g. $\mathbf{r}, \mathbf{T}, \mathbf{G}$
- some inputs are endogenous **unknowns**, e.g. $\mathbf{Y}$
- some outputs are **targets** that must be zero in GE, e.g. $\mathbf{H}$ [$\# \text{targets} = \# \text{unknowns}$]

Most macro models can be written this way. Will help us solve them efficiently!

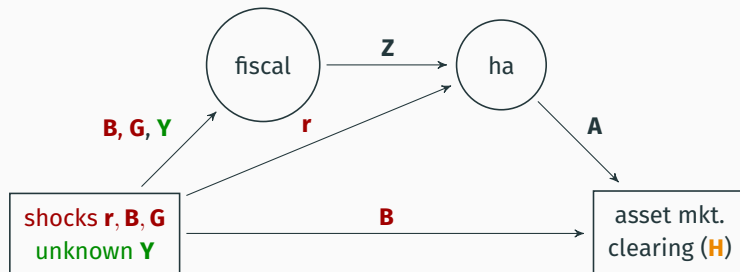
Require that models have no cycles $\rightarrow$ draw as directed acyclic graphs (DAGs).



- Model is composite mapping: $(Y, r, T, G) \rightarrow H$.
- GE response of Y to shocks satisfies $H = 0$.

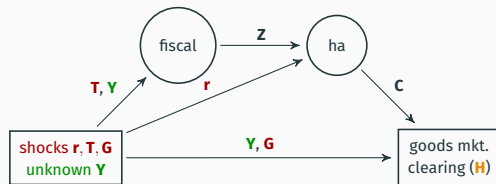
Side note: DAGs are not unique

- E.g. instead of feeding in **G** and **T** shocks, could feed in **G** and **B** shocks
- Could use asset market rather than goods market clearing



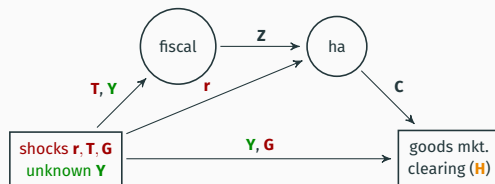
- We'll use this approach later in the tutorial.

Solving for output response to shocks



- Imagine we change the path of government spending G . How is Y affected?

Solving for output response to shocks



- Imagine we change the path of government spending G . How is Y affected?
- We need to find Y such that

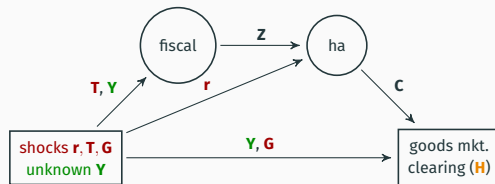
$$H(Y, G) = 0$$

- First order shock $dG \Rightarrow$ use implicit function theorem:

$$dY = - (H_Y)^{-1} \cdot H_G \cdot dG$$

All we need is H 's Jacobians H_Y and H_G ...

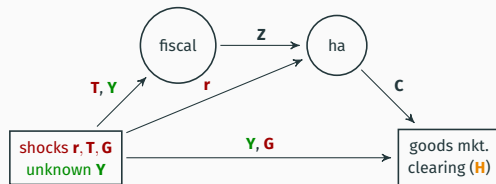
How do we get the Jacobians?



First step is to compute **individual blocks' Jacobians**, e.g. $\mathcal{J}^{\mathbf{z}, \mathbf{Y}}$, $\mathcal{J}^{\mathbf{c}, \mathbf{z}}$, $\mathcal{J}^{\mathbf{H}, \mathbf{Y}}$, $\mathcal{J}^{\mathbf{H}, \mathbf{G}}$

- If block is analytical (SimpleBlock), its derivative is analytical too
 - e.g. $\mathcal{J}^{\mathbf{z}, \mathbf{Y}} = \mathbf{I}$ or $\mathcal{J}^{\mathbf{H}, \mathbf{G}} = \mathbf{I}$
- If block has heterogeneous agents (HetBlock), solve Jacobian numerically
 - e.g. solve $\mathcal{J}^{\mathbf{c}, \mathbf{z}}$ using fake news algorithm

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Then “chain” the Jacobians together:

$$\mathbf{H}_{\mathbf{Y}} = \mathcal{J}^{\mathbf{H}, \mathbf{Y}} + \mathcal{J}^{\mathbf{H}, \mathbf{C}} \cdot \mathcal{J}^{\mathbf{C}, \mathbf{Z}} \cdot \mathcal{J}^{\mathbf{Z}, \mathbf{Y}} \quad \mathbf{H}_{\mathbf{G}} = \mathcal{J}^{\mathbf{H}, \mathbf{G}}$$

SSJ workflow (will use this many times!)

These ideas are at the heart of the workflow in our Sequence-Space Jacobian toolbox:

1. Define individual blocks: SimpleBlock, HetBlock, SolvedBlock
 - SolvedBlock allows to solve out recursions, e.g. solve an Euler equation
2. Combine the blocks into a model
3. Set steady state parameters and solve the model at the steady state.
4. Solve for the responses of the model directly, code handles all Jacobians.
 - e.g. solve_impulse_linear automatically computes $d\mathbf{Y} = -(\mathbf{H}_Y)^{-1} \cdot \mathbf{H}_G \cdot d\mathbf{G}$
 - but can also compute $\mathbf{H}_Y$, $\mathbf{H}_G$ individually, or even $\mathcal{J}^{C,Z}$, $\mathcal{J}^{Z,Y}$ etc
 - this will be helpful to inspect the model's mechanics!

Summary

We introduced a canonical HANK model:

- Standard incomplete markets households
- Standard New-Keynesian supply side, but sticky wages + flex prices
- Real rate rule for now (relax later)

Outlined how we can solve this model ...

- Set up as blocks. Many blocks = a model
- SSJ toolbox solves out Jacobians, chains them, uses implicit function theorem to compute IRFs

Next: Analyze fiscal policy in this model. Tomorrow: Monetary policy.

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