# The canonical HANK model

Goethe Heterogeneous-Agent Macro Workshop

Ludwig Straub

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## This session

Just saw: How to solve steady states and PE transitional dynamics of neoclassical heterogeneous agent models.

**Next:** Introducing "HANK".

1. The canonical HANK model
2. Three instructive special cases
3. Solving the model using blocks and DAGs
4. Summary

# The canonical HANK model

# Introducing the canonical HANK model

- We now embed the standard incomplete markets model of consumption and saving into a New-Keynesian model
- Along the way, we will allow for a **government**: bonds, taxes, gov. spending
- Will mostly follow Auclert et al. (2024) (henceforth IKC), though allowing for monetary policy, too
- Will set up the model in the **sequence space**:
  - assume economy in steady state, feed in perfect foresight aggregate shocks at $t = 0$, study response thereafter ("MIT shocks")
  - keep in mind **certainty equivalence** !!

# Household side

- Household $i$ solves

$$\max_{c_{it}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{it}) - v\left(n_{it}\right)\right)$$

$$c_{it} + a_{it} \leq (1 + r_t)a_{it-1} + z_{it}$$

$$a_{it} \geq \underline{a}$$

- This is the "sequence" analogue of the Bellman equation from before
- Real after tax income is

$$z_{it} = (1 - \tau_t)y_{it} \qquad\qquad y_{it} = \frac{W_t}{P_t} e_{it} n_{it}$$

- Here, $e_{it}$ is idiosyncratic productivity, normalized so that $\mathbb{E}_i\left[e_{it}\right] = 1$
- Can capture progressive taxation as in Heathcote et al. (2017) (see IKC paper)

# Unions and sticky wages

- For our canonical HANK model, we'll work with **sticky wages** (not prices)
  - with sticky prices, can get countercyclical profits
  - ... redistribution from wage to profit earners in recession... matters in HANK!
  - ... strange results can happen (examples: Bilbiie 2008, Broer et al. 2020)
- Microfound sticky wages extending Erceg et al. (2000) (see IKC)
  - + "labor allocation rule": which agent works what fraction of total labor $N_t$?
  - today: assume all agents work same hours, $n_{it} = N_t$
- **Today:** will use simple ad-hoc wage Phillips curve (details won't matter)

$$\pi_t^w = \kappa \underbrace{\left( v'\left(N_t\right) - \frac{\epsilon - 1}{\epsilon}\left(1 - \tau_t\right) \frac{W_t}{P_t} u'\left(C_t\right)\right)}_{\text{wedge in labor FOC of "average" agent}} + \beta \pi_{t+1}^w$$

# Production

- Representative firm with aggregate production function, linear in labor

$$Y_t = X_t N_t$$

  where $X_t$ is TFP

- Assume **flexible prices** $\Rightarrow$

$$P_t = \frac{W_t}{X_t} \qquad \Leftrightarrow \qquad \frac{W_t}{P_t} = X_t$$

  Real wage is exogenous. No profits!

- Goods inflation $\pi_t$ = wage inflation $\pi_t^w$ minus TFP growth

# Government: Fiscal policy

Government sets fiscal policy, consisting of paths

- $G_t$ of gov spending
- $T_t$ of total tax revenue, controlled via $\tau_t$

$$T_t = \tau_t Y_t$$

- $B_t$ of government bonds, uniformly bounded (no Ponzi schemes)

subject to government budget constraint

$$B_t = (1 + r_t) B_{t-1} + G_t - T_t$$

Can set any two of those and the third follows.

# Government: How is after tax income distributed?

- Total after-tax income is

$$Z_t \equiv Y_t - T_t = (1 - \tau_t) Y_t$$

- Because of linear taxes, we find for individual after-tax income

$$y_{it} = e_{it} Y_t \qquad \Rightarrow \qquad z_{it} = e_{it} Z_t$$

- $z_{it}$ simply a share of total after-tax income $Z_t$. Will be convenient.

# Government: Monetary policy

Monetary authority follows an interest rate rule. Allow for two kinds of rules:

- **standard Taylor rule.** (linearized)

$$i_t = r + \phi_\pi \pi_t + \epsilon_t$$

  here: $r =$ steady state real rate, $\epsilon_t =$ monetary shock

- **real rate rule.**

$$r_{t+1} = r + \epsilon_t \qquad \Leftrightarrow \qquad i_t = r + \pi_{t+1} + \epsilon_t$$

  Equivalent to Taylor rule with coefficient 1 on expected inflation.
  Note: $\pi_{t+1}$ vs $\pi_t$ not key (same in cts time!), key is $\phi_\pi = 1$

Why allow for "real rate rule"? Huge gain in tractability! All monetary policy acts via changing real rate. Cost is small if Phillips curve is flat ($\pi_t$ moves little).

# Definition of equilibrium

- All agents optimize and markets clear

$$G_t + C_t = Y_t$$
$$A_t = B_t$$

where household aggregates are

$$C_t = \int c_t^* \left(a_-, e\right) dD_t \left(a_-, e\right)$$

$$A_t = \int a_t^* (a_-, e) dD_t \left(a_-, e\right)$$

# Computing the steady state

How can we find the steady state of this model?

1. Normalize $Y = 1$, calibrate $r$ and $B, G$. Set $T = G + rB$.
2. Can use **same code** as for standard incomplete markets model:
   - instead of $e_{it}Y$ now use $e_{it} \cdot (Y - T)$
   - choose $\beta$ to match $A = B$.
3. $G + C = Y$ holds by Walras law! Done!

# Three instructive special cases

# Special cases

Will introduce three special cases that are helpful to analyze and compare the HA model to.

1. Representative-agent model (RA) — [Woodford 2003, Galí 2008]
2. Two-agent model (TA) — [Campbell and Mankiw 1989, Galí et al. 2007, Bilbiie 2008]
3. Zero-liquidity model (ZL) — [Werning 2015, Ravn and Sterk 2017, Bilbiie 2021]

Only difference across models: how $C_t$ is determined given real rates $r_t$ and after-tax incomes $Z_t$.

Steady state aggregates are identical across models.

# Representative-agent model

- This is the standard NK model (with wage rigidities)
- Consumption solves

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

$$C_t + a_t \leq (1+r_t)a_{t-1} + Z_t$$

which has the solution

$$C_t = \frac{\beta^{t/\sigma} q_t^{-1/\sigma}}{\sum_{s\geq 0} \beta^{s/\sigma} q_s^{1-1/\sigma}} \left[ \sum_{s\geq 0} q_s Z_s + (1+r_0)a_{-1} \right]$$

where $q_t \equiv (1+r_1)^{-1} \cdot \cdots \cdot (1+r_t)^{-1}$. With $r_t = r = \beta^{-1} - 1$, this is just

$$C_t = \frac{r}{1+r} \sum_{s\geq 0} (1+r)^{-s} Z_s + r a_{-1}$$

# Two-agent model

- This is like an RA economy except that a fraction $\mu$ is hand-to-mouth (HTM). Only $1-\mu$ behave according to PIH.
- PIH agents' consumption is determined by

$$c_t^{PIH} = \frac{\beta^{t/\sigma} q_t^{-1/\sigma}}{\sum_{s\geq 0} \beta^{s/\sigma} q_s^{1-1/\sigma}} \left[ \sum_{s\geq 0} q_s Z_s + (1+r_0) a_{-1} \right]$$

- HTM agents' consumption is determined by

$$c_t^{HTM} = Z_t$$

- Jointly pin down aggregate consumption

$$C_t = (1-\mu) c_t^{PIH} + \mu c_t^{HTM}$$

# Zero-liquidity model

- Assume $\underline{a} = 0$
- What if we shrink liquidity $B_t$ down to $\underline{a} = 0$? (e.g. via smaller $\beta$)
- Eventually, **all** agents must have zero assets, so $c_{it} = z_{it} = e_{it} Z_t$
- Does that mean all Euler equations fail? Consider:

$$z_{it}^{-\sigma} \geq \beta(1 + r_{t+1})\mathbb{E}_t\left[z_{it+1}^{-\sigma}\right] \qquad \Leftrightarrow \qquad Z_t^{-\sigma} \geq \beta(1 + r_{t+1})\underbrace{\mathbb{E}\left[\frac{(e')^{-\sigma}}{e^{-\sigma}}\middle| e\right]}_{\rho(e)} Z_{t+1}^{-\sigma}$$

- The last Euler equation to fail as we reduce $\beta$ is that of $\overline{e} = \arg\max \rho(e)$
- Tractable! For instance, steady state $r$ solves $\beta(1 + r)\rho(\overline{e}) = 1$

# Solving the model using blocks and DAGs

# Blocks

- Throughout this workshop, we will see that it is very useful to break models into "blocks"
- This language is often loosely used in practice, we will formally define them
  - reference is Auclert et al. (2021)
- We will write sequences of variables, e.g. $\{r_t\}$, as vectors $\mathbf{r} = (r_0, r_1, \ldots)'$.

# Defining blocks and models

**Block:** mapping from sequences of *inputs* to sequences of *outputs*.

Examples:

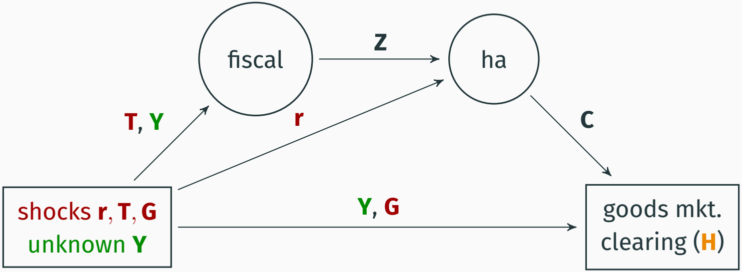
- Household block: $\mathbf{r}, \mathbf{Z} \rightarrow \mathbf{C}, \mathbf{A}$
- Fiscal policy block: $\mathbf{r}, \mathbf{T}, \mathbf{G}, \mathbf{Y} \rightarrow \mathbf{B}, \mathbf{Z}$
- Goods market clearing block: $\mathbf{Y}, \mathbf{C}, \mathbf{G} \rightarrow \mathbf{H} \equiv \mathbf{C} + \mathbf{G} - \mathbf{Y}$

**Model:** combination of blocks

- some inputs are exogenous **shocks**, e.g. $\mathbf{r}, \mathbf{T}, \mathbf{G}$
- some inputs are endogenous **unknowns**, e.g. $\mathbf{Y}$
- some outputs are **targets** that must be zero in GE, e.g. $\mathbf{H}$ [#targets = #unknowns]

Most macro models can be written this way. Will help us solve them efficiently!

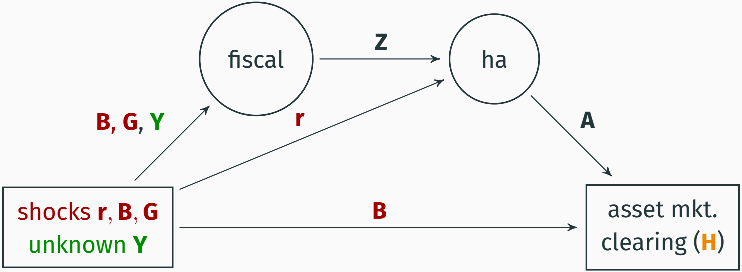
Require that models have no cycles $\rightarrow$ draw as directed acyclic graphs (DAGs).




- Model is composite mapping: $(\mathbf{Y}, \mathbf{r}, \mathbf{T}, \mathbf{G}) \rightarrow \mathbf{H}$.
- GE response of $\mathbf{Y}$ to shocks satisfies $\mathbf{H} = 0$.

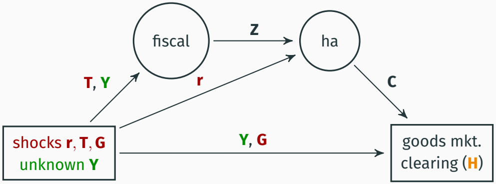
# Side note: DAGs are not unique

- E.g. instead of feeding in **G** and **T** shocks, could feed in **G** and **B** shocks
- Could use asset market rather than goods market clearing




- We'll use this approach later in the tutorial.

# Solving for output response to shocks



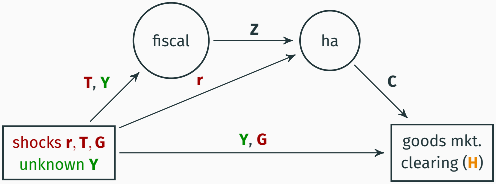

- Imagine we change the path of government spending **G**. How is **Y** affected?
- We need to find **Y** such that

$$\mathbf{H}(\mathbf{Y}, \mathbf{G}) = 0$$

- First order shock $d\mathbf{G} \Rightarrow$ use implicit function theorem:

$$d\mathbf{Y} = -\left(\mathbf{H}_{\mathbf{Y}}\right)^{-1} \cdot \mathbf{H}_{\mathbf{G}} \cdot d\mathbf{G}$$

All we need is **H**'s Jacobians $\mathbf{H}_{\mathbf{Y}}$ and $\mathbf{H}_{\mathbf{G}}$ ...




First step is to compute **individual blocks' Jacobians**, e.g. $\mathcal{J}^{\mathbf{Z},\mathbf{Y}}$, $\mathcal{J}^{\mathbf{C},\mathbf{Z}}$, $\mathcal{J}^{\mathbf{H},\mathbf{Y}}$, $\mathcal{J}^{\mathbf{H},\mathbf{G}}$

- If block is analytical (`SimpleBlock`), its derivative is analytical too
  - e.g. $\mathcal{J}^{\mathbf{Z},\mathbf{Y}} = \mathbf{I}$ or $\mathcal{J}^{\mathbf{H},\mathbf{G}} = \mathbf{I}$
- If block has heterogeneous agents (`HetBlock`), solve Jacobian numerically
  - e.g. solve $\mathcal{J}^{\mathbf{C},\mathbf{Z}}$ using fake news algorithm

Then "chain" the Jacobians together:

$$\mathbf{H}_{\mathbf{Y}} = \mathcal{J}^{\mathbf{H},\mathbf{Y}} + \mathcal{J}^{\mathbf{H},\mathbf{C}} \cdot \mathcal{J}^{\mathbf{C},\mathbf{Z}} \cdot \mathcal{J}^{\mathbf{Z},\mathbf{Y}} \qquad \mathbf{H}_{\mathbf{G}} = \mathcal{J}^{\mathbf{H},\mathbf{G}}$$

# SSJ workflow (will use this many times!)

These ideas are at the heart of the workflow in our Sequence-Space Jacobian toolbox:

1. Define individual blocks: `SimpleBlock`, `HetBlock`, `SolvedBlock`
   - `SolvedBlock` allows to solve out recursions, e.g. solve an Euler equation
2. Combine the blocks into a model
3. Set steady state parameters and solve the model at the steady state.
4. Solve for the responses of the model directly, code handles all Jacobians.
   - e.g. `solve_impulse_linear` automatically computes $d\mathbf{Y} = -\left(\mathbf{H_Y}\right)^{-1} \cdot \mathbf{H_G} \cdot d\mathbf{G}$
   - but can also compute $\mathbf{H_Y}$, $\mathbf{H_G}$ individually, or even $\mathcal{J}^{\mathbf{C},\mathbf{Z}}$, $\mathcal{J}^{\mathbf{Z},\mathbf{Y}}$ etc
   - this will be helpful to inspect the model's mechanics!

# Summary

## Summary

We introduced a canonical HANK model:

- Standard incomplete markets households
- Standard New-Keynesian supply side, but sticky wages + flex prices
- Real rate rule for now (relax later)

Outlined how we can solve this model ...

- Set up as blocks. Many blocks $=$ a model
- SSJ toolbox solves out Jacobians, chains them, uses implicit function theorem to compute IRFs

**Next:** Analyze fiscal policy in this model. Tomorrow: Monetary policy.

# References

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